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LETTER TO THE EDITOR

COMMENT ON 'APPLICATION OF GENERALIZED QUADRATURE TO SOLVE TWO-DIMENSIONAL INCOMPRESSIBLE NAVIER–STOKES EQUATIONS', BY C. SHU AND B. E. RICHARDS

Referring to the paper by Shu and Richards which appeared in *Int j. numer. methods fluids*,[1] I would like to bring to attention that the formulae for the generalized differential quadrature and for calculation of the weighting coefficients have previously been reported in the literature. In addition, many other developments have occurred following Bellman *et al.*[2]

Civan and Sliepcevich[3–6] extended and generalized the method of differential quadrature for approximation of higher-order partial derivatives of single and multivariable functions. Civan and Sliepcevich[7] extended the quadrature method for integro-differential equations. Civan and Sliepcevich[7,8] introduced the concept of approximating composite and intricate operations by a single quadrature. Blick and Civan[9] and Civan[10] demonstrated that the application of differential quadrature to highly non-linear phenomena leads to very practical and accurate computational schemes. Civan[11] extended the differential quadrature method to differential cubature for multivariable functions. Civan[12] developed a generalized quadrature and cubature approach for integrals, derivatives and composite operators for multivariable functions, and presented applications in which both the spatial and temporal derivatives and integrals were approximated by these methods for the first time, to the author's knowledge. Civan[12,13] has shown that the cubature method is particularly advantageous over the quadrature method when mixed operations, such as $\partial^2/\partial x\partial y$ and $(\partial/\partial x)\int \partial y$, were involved. Civan[11–13] has also developed a generalized methodology for determination of the quadrature and cubature weighting coefficients for general applications.

Quan and Chang[14] developed the explicit formulae for calculation of the differential quadrature weights using the Lagrange interpolation process and the Legendre, ultraspherical and Chebyshev polynomials. Civan[11] pointed out that the ill-conditioning problem associated with the Vandermonde matrix can be alleviated by using the Björk and Pereyra[15] method. The aforementioned works and many others published in the literature including those by Jang *et al.*[16] and Sherbourne and Pandey[17] as well as Shu and Richards[1] are the applications of the differential quadrature method which indicate that the quadrature leads to accurate results with less computational effort. Civan[12,13] demonstrates that the cubature approach can achieve better results for multivariate functions. Civan[11] shows that the cubature method can be applied for local discretization which is also true for the quadrature.

The quadrature and cubature methods form a practical basis for the development of a variety of numerical schemes (including some of the commonly known formulae) by choosing different functional representations including the conventional polynomials and those derived from the analytical solutions of the locally linearized equations[18] to determine the quadrature and cubature weights. Therefore, these are highly versatile innovative methods which promise a great potential for future research.

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